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LETTER TO THE EDITOR

# New shape-invariant potentials in supersymmetric quantum mechanics

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**Abstract.** Quantum mechanical potentials satisfying the property of shape invariance are well known to be algebraically solvable. Using a scaling ansatz for the change of parameters, we obtain a large class of new shape-invariant potentials which are reflectionless and possess an infinite number of bound states. They can be viewed as $q$-deformations of the single soliton solution corresponding to the Rosen–Morse potential. Explicit expressions for energy eigenvalues, eigenfunctions and transmission coefficients are given. Included in our potentials as a special case is the self-similar potential recently discussed by Shabat and Spiridonov.

In recent years, supersymmetric quantum mechanics [1] has yielded many interesting results. Some time ago, Gendenshtein pointed out that supersymmetric partner potentials satisfying the property of shape invariance and unbroken supersymmetry are exactly solvable [2]. The shape invariance condition is

$$V_+(x, a_0) = V_-(x, a_1) + R(a_0) \tag{1}$$

where $a_0$ is a set of parameters and $a_1 = f(a_0)$ is an arbitrary function describing the change of parameters. The common $x$-dependence in $V_-$ and $V_+$ allows full determination of energy eigenvalues [2], eigenfunctions [3] and scattering matrices [4], algebraically. One finds ($\hbar = 2m = 1$)

$$E_n^{(-)}(a_0) = \sum_{k=0}^{n-1} R(a_k) \qquad E_0^{(-)}(a_0) = 0 \tag{2}$$

$$\begin{aligned} \psi_n^{(-)}(x, a_0) &= \left[-\frac{\mathrm{d}}{\mathrm{d}x} + W(x, a_0)\right] \psi_{n-1}^{(-)}(x, a_1) \\ \psi_0^{(-)}(x, a_0) &\propto \exp\left(-\int^x W(y, a_0)\,\mathrm{d}y\right) \end{aligned} \tag{3}$$

where the superpotential $W(x, a_0)$ is related to $V_\pm(x, a_0)$ by

$$V_\pm(x, a_0) = W^2(x, a_0) \pm W'(x, a_0). \tag{4}$$

In terms of $W$, the shape invariance condition reads

$$W^2(x, a_0) + W'(x, a_0) = W^2(x, a_1) - W'(x, a_1) + R(a_0). \tag{5}$$

It is still a challenging open problem to identify and classify the solutions to (5). Certain

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solutions to the shape invariance condition are known [5]. (They include the harmonic oscillator, Coulomb, Morse, Eckart and Poschl–Teller potentials.) In all these cases, it turns out that $a_1$ and $a_0$ are related by a translation ($a_1 = a_0 + \alpha$). Careful searches with this ansatz have failed to yield any additional shape-invariant potentials [6]. Indeed it has been suggested [7] that there are no other shape-invariant potentials. Although a rigorous proof has never been presented, no counterexamples have so far been found either.

In this letter, we consider solutions of (5) resulting from a new scaling ansatz

$$a_1 = qa_0 \tag{6}$$

where $0 < q < 1$. This choice was motivated by the recent interest in $q$-deformed Lie algebras. It enables us to find a large class of new shape-invariant potentials, all of which are reflectionless and possess an infinite number of bound states. As a special case, our approach includes the self-similar potential studied by Shabat [8] and Spiridonov [9].

Consider an expansion of the superpotential of the form

$$W(x, a_0) = \sum_{j=0}^{\infty} g_j(x) a_0^j . \tag{7}$$

Substituting into (5), writing $R(a_0)$ in the form

$$R(a_0) = \sum_{j=0}^{\infty} R_j a_0^j \tag{8}$$

and equating powers of $a_0$ yields

$$g_0'(x) = \frac{R_0}{2} \qquad g_0(x) = \frac{R_0 x}{2} + C_0 \tag{9}$$

$$g_n'(x) + 2d_n g_0(x) g_n(x) = d_n r_n - d_n \sum_{j=1}^{n-1} g_j(x) g_{n-j}(x) \tag{10}$$

where

$$R_n \equiv (1-q^n) r_n \qquad d_n \equiv \frac{1-q^n}{1+q^n} \qquad (n = 1, 2, 3 \ldots). \tag{11}$$

This set of linear differential equations is easily solvable in succession yielding a general solution of (5). Note that the limit $q \to 0$ is particularly simple yielding the one-soliton Rosen–Morse potential of the form $W = \gamma \tanh \gamma x$. Thus our results can be regarded as multiparameter deformations of this potential corresponding to different choices of $R_n$. For simplicity, in this letter we shall confine our attention to the special case $g_0(x) = 0$ (i.e. $R_0 = C_0 = 0$), while the more general case will be discussed elsewhere [10].

*For* $g_0 = 0$, *the solution is*

$$g_n(x) = d_n \int \mathrm{d}x \left[ r_n - \sum_{j=1}^{n-1} g_j(x) g_{n-j}(x) \right]. \tag{12}$$

For the simplest case $r_n = 0$, $n \geqslant 2$ we obtain the superpotential $W$ as given by Shabat [8] and Spiridonov [9], provided we choose $d_1 r_1 a_0 = \gamma^2$ and replace $q$ by $q^2$. This shows that the self-similarity condition of these authors is, in fact, a special case of the shape invariance condition (5). This comment is also true in case any one $r_n$ (say $r_j$) is taken to be non-zero and $q^j$ is replaced by $q^2$.

Let us now consider a somewhat more general case when $r_n=0$, $n\geqslant 3$. Using (12) we can readily calculate all $g_n(x)$. The first three are

$$g_1(x)=d_1r_1x \qquad g_2(x)=d_2r_2x-\tfrac{1}{3}d_1^2r_1^2d_2x^3$$
$$g_3(x)=-\tfrac{2}{3}d_1r_1d_2r_2d_3x^3+\tfrac{2}{15}d_1^3r_1^3d_2d_3x^5. \tag{13}$$

Note that $W(x)$ contains only odd powers of $x$. This makes the potential $V_-(x)$ symmetric in $x$ and also guarantees the situation of unbroken supersymmetry. The energy eigenvalues follow immediately from (2) and (8) ($n=0, 1, 2, \ldots, \infty; 0<q<1$)

$$E_n^{(-)}(a_0)=R_1a_0\frac{(1-q^n)}{(1-q)}+R_2a_0^2\frac{(1-q^{2n})}{(1-q^2)}. \tag{14}$$

The superpotential $W(x)$ can be written in a somewhat more compact form if one defines $\gamma_1^2\equiv d_1r_1a_0$ and $\gamma_2^2\equiv d_2r_2a_0^2$. The unnormalized ground-state wavefunction is easily obtained from (3):

$$\psi_0^{(-)}(x, a_0)=\exp\left[-\frac{x^2}{2}(\gamma_1^2+\gamma_2^2)+\frac{x^4}{12}(d_2\gamma_1^4+2d_3\gamma_1^2\gamma_2^2+d_4\gamma_2^4)+0(x^6)\right]. \tag{15}$$

The excited-state wavefunctions can be recursively calculated by using (3) with $a_1=qa_0$.

The transmission coefficient of two symmetric partner potentials are related by [11]

$$T_-(k, a_0)=\left[\frac{ik-W(\infty, a_0)}{ik+W(\infty, a_0)}\right]T_+(k, a_0) \tag{16}$$

where $k=[E-W^2(\infty, a_0]^{1/2}$. For a shape-invariant potential

$$T_+(k, a_0)=T_-(k, a_1). \tag{17}$$

Repeated application of (16) and (17) gives

$$T_-(k, a_0)=\left[\frac{ik-W(\infty, a_0)}{ik+W(\infty, a_0)}\right]\left[\frac{ik-W(\infty, a_1)}{ik+W(\infty, a_1)}\right]\ldots\left[\frac{ik-W(\infty, a_{n-1})}{ik+W(\infty, a_{n-1})}\right]T_-(k, a_n) \tag{18}$$

where

$$W(\infty, a_j)=(E_\infty^{(-)}-E_j^{(-)})^{1/2}. \tag{19}$$

As $n\to\infty$, since $a_n=q^na_0$ and we have taken $g_0(x)=0$, one obtains $W(x, a_n)\to 0$. This corresponds to a free particle for which the transmission coefficient is unity. Thus, for the potential $V_-(x, a_0)$, the reflection coefficient vanishes and the transmission coefficient is

$$T_-(k, a_0)=\prod_{j=0}^{\infty}\left[\frac{ik-W(\infty, a_j)}{ik+W(\infty, a_j)}\right]. \tag{20}$$

Clearly, $|T|^2=1$ and the poles of $T_-$ correspond to the energy eigenvalues of (14). Note that one does not get reflectionless potentials for the case $g_0(x)\neq 0$. This will be further discussed in [10].

The above discussion, keeping only $r_1, r_2 \neq 0$, can readily be generalized to an arbitrary number of non-zero $r_j$. The energy eigenvalues for this case are given by

$$E_n^{(-)}(a_0) = \sum_j R_j a_0^j \left( \frac{1-q^{jn}}{1-q^j} \right) \qquad n = 0, 1, 2, \ldots \tag{21}$$

All of these potentials are also reflectionless with $T_-$ as given by (19)–(21). One expects that these symmetric reflectionless potentials can also be derived using previously developed methods [12] and the spectrum given in (21).

In (7) and (8) we have only kept positive powers of $a_0$. If, instead, we had only kept negative powers of $a_0$, then the spectrum would be similar except that one has to choose the deformation parameter $q > 1$. However, a mixture of positive and negative powers of $a_0$ is not allowed in general since neither $q < 1$ nor $q > 1$ will give an acceptable spectrum. For the enlarged class of shape-invariant potentials discussed in this letter, it is clear [3] that the lowest order supersymmetric WKB approximation [13] will yield the exact spectrum.

We conclude with two brief remarks on extensions of the work described in this letter. We have been able to construct new shape-invariant potentials which are $q$-deformations of the potentials corresponding to multi-soliton systems [10]. Also, it is possible to show [10] that with the choice $g_0(x) \neq 0$, one obtains $q$-deformations of the one-dimensional harmonic oscillator potential.

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